Calc. Var. (2007) 28:427–447
DOI 10.1007/s00526-006-0048-7

Calculus of Variations

A chain rule formula in BV and application to lower semicontinuity

Virginia De Cicco · Nicola Fusco · Anna Verde

Received: 25 June 2004 / Accepted: 3 June 2005 /
Published online: 16 August 2006

1 Introduction

It is well known that if u is a BV function from a bounded open set $\Omega \subset \mathbb{R}^N$ and $B : \mathbb{R} \to \mathbb{R}$ is Lipschitz, the composition $v = B \circ u$ is also in $BV(\Omega)$ and the following chain rule formula holds

$$Dv = B'(\widetilde{u})\nabla u \mathcal{L}^N + B'(\widetilde{u})D^c u + (B(u^+) - B(u^-))\nu_u \mathcal{H}^{N-1} \llcorner J_u,$$

where ∇u is the absolutely continuous part of $Du, D^c u$ is the Cantor part of Du and J_u is the jump set of u (for the definition of these and other relevant quantities, see Sect. 2). A delicate issue about this formula concerns the meaning of the first two terms on the right hand side. In fact, in order to understand why they are well defined, one has to take into account that $B'(t)$ exists for $\mathcal{L}^1$-a.e. t and that, if E is an $\mathcal{L}^1$-null set in $\mathbb{R}$, not only ∇u vanishes $\mathcal{L}^N$-a.e. on $\widetilde{u}^{-1}(E)$, but also $|D^c u|(\widetilde{u}^{-1}(E)) = 0$ (see [2, Theorem 3.92]). The difficulty of giving a correct meaning to the various parts in which the derivative of a BV function can be split is even greater when u is a vector field, a case where a chain rule formula has been proved by Ambrosio and Dal Maso in [1]. In particular, their result applies to the composition of a scalar BV function with a Lipschitz function B depending also on x, namely to the function $B(x, u(x))$, where $B : \Omega \times \mathbb{R} \to \mathbb{R}$ is Lipschitz.

V. De Cicco
Dipartimento di Metodi e Modelli, Matematici per le Scienze Applicate,
Università di Roma "La Sapienza", Via Scarpa 16,
Rome 00161, Italy

N. Fusco (✉) · A. Verde
Dipartimento di Matematica e Applicazioni "R. Caccioppoli",
Università degli Studi di Napoli "Federico II",
via Cinzia Complesso Universitario Monte S. Angelo,
Napoli 80126, Italy
e-mail: n.fusco@unina.it

In many applications, however, B has the special form

$$B(x,t)=\int_0^t b(x,s)\mathrm{d}s, \tag{1.1}$$

but, on the other hand, one would like to assume only a weak differentiability of B with respect to x (or even less). In this spirit, De Cicco and Leoni ([6]) have obtained a chain rule formula in the case $B(x,t)$ is a vector field such that $\mathrm{div}_x B(\cdot,t)$ belongs to $L^1(\Omega)$, uniformly with respect to t, and u is in $W^{1,1}(\Omega)$. In the same paper they prove an L^1-lower semicontinuity result in $W^{1,1}$ by applying their formula to a vector field B of the type (1.1).

In this paper we extend these results to the case where u is a BV function and replace the assumption that $\mathrm{div}_x B$ is in L^1 with a BV dependence with respect to x. Namely, we prove the following

Theorem 1.1 *Let $\Omega \subset \mathrm{I\!R}^N$ be an open bounded set and let $b:\Omega\times\mathrm{I\!R}\to\mathrm{I\!R}$ be a locally bounded Borel function. Assume that*

(i) *for $\mathcal{L}^1$-a.e. $t\in\mathrm{I\!R}$ the function $b(\cdot,t)\in BV(\Omega)$;*
(ii) *for any compact set $H\subset\mathrm{I\!R}$,*

$$\int_H |D_x b(\cdot,t)|(\Omega)\,\mathrm{d}t<+\infty.$$

Then, for every $u\in BV(\Omega)\cap L^\infty_{loc}(\Omega)$, the function $v:\Omega\to\mathrm{I\!R}$, defined by

$$v(x):=\int_0^{u(x)} b(x,t)\,\mathrm{d}t,$$

belongs to $BV_{loc}(\Omega)$ and for any $\phi\in C^1_0(\Omega)$ we have

$$\begin{aligned}\int_\Omega \nabla\phi(x)v(x)\,\mathrm{d}x = &-\int_{-\infty}^{+\infty}\mathrm{d}t\int_\Omega \mathrm{sgn}(t)\ \chi^*_{\Omega_{u,t}}(x)\phi(x)\,\mathrm{d}D_x b(x,t)\\ &-\int_\Omega \phi(x)b^*(x,\widetilde{u}(x))\nabla u(x)\,\mathrm{d}x\\ &-\int_\Omega \phi(x)b^*(x,\widetilde{u}(x))\,\mathrm{d}D^c u(x)\\ &-\int_{J_u}\phi(x)\nu_u(x)\,\mathrm{d}\mathcal{H}^{N-1}(x)\int_{u^-(x)}^{u^+(x)} b^*(x,t)\,\mathrm{d}t,\end{aligned} \tag{1.2}$$

where $\Omega_{u,t}=\{x\in\Omega:\ t$ belongs to the segment of endpoints 0 and $u(x)\}$ and $\chi^*_{\Omega_{u,t}}$ and $b^*(\cdot,t)$ are, respectively, the precise representatives of $\chi_{\Omega_{u,t}}$ and $b(\cdot,t)$.

Notice that all the integrals on the right hand side of (1.2) are well defined. In fact $b^*(x,t)$ is a locally bounded Borel function, $\widetilde{u}$ is Borel, hence $b^*(x,\widetilde{u}(x))$ is a Borel function too. Similarly, the function $(x,t)\in\Omega\times\mathrm{I\!R}\to\chi^*_{\Omega_{u,t}}(x)$ is a Borel function, hence it makes sense to integrate it first with respect to the vector measure $D_x b(\cdot,t)$ and then with respect to t.

To prove (1.2) we start by regularizing b with respect to x, so to get a Lipschitz approximation to which the Ambrosio and Dal Maso chain rule formula applies. Then, the rest of the proof consists in analyzing carefully the convergence of all the terms in (1.2), those involving the various parts of the derivative of u and the one containing the derivative of b with respect to x. Each of these terms requires a different argument. We notice also that, when dealing with a function u in $W^{1,1}$, the assumptions of Theorem 1.1 can be weakened by considering a vector field $b : \Omega \times \mathbb{R} \to \mathbb{R}^N$ such that $\mathrm{div}_x b(\cdot,t)$ is a Radon measure for $\mathcal{L}^1$-a.e. t. At this regard our Theorem 3.4 can be viewed as a generalization of the afore mentioned result proved in [6].

Let us now turn to the application of (1.2) to lower semicontinuity. Recent papers by Fonseca and Leoni, Gori, Maggi and Marcellini, and by the authors of this paper (see [6,7,11,12,14,15]) have shown that the classical conditions due to Serrin [16], ensuring the L^1-lower semicontinuity in $W^{1,1}$ of a functional of the type

$$F(u) = \int_\Omega f(x,u(x),\nabla u(x))\,\mathrm{d}x, \quad u \in W^{1,1}(\Omega), \tag{1.3}$$

can be considerably weakened. In particular, Gori et al. in [15] show that beside the usual convexity assumption with respect to the gradient and the continuity with respect to u, in order to prove that F is lower semicontinuous it is enough to assume that f is (uniformly) weakly differentiable in x, namely that for any compact set $H \subset \mathbb{R} \times \mathbb{R}^N$

$$\int_\Omega |\nabla_x f(x,t,\xi)|\,\mathrm{d}x \le L \quad \text{for every } (t,\xi) \in H,$$

for some constant $L \equiv L(H)$. As a consequence of Theorem 1.1, we are able to improve their result by replacing the weak differentiabilty of f with respect to x with a BV dependence on x.

Theorem 1.2 *Let us assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,+\infty)$ is a locally bounded Borel function such that*

$$f(x,t,\cdot) \text{ is convex in } \mathbb{R}^N \text{ for every } (x,t) \in \Omega \times \mathbb{R}, \tag{1.4}$$

$$f(x,\cdot,\xi) \text{ is continuous in } \mathbb{R} \text{ for every } (x,\xi) \in \Omega \times \mathbb{R}^N, \tag{1.5}$$

$$f(\cdot,t,\xi) \in BV(\Omega) \text{ for every } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.6}$$

and such that, for any compact $K \subset \mathbb{R} \times \mathbb{R}^N$, there exists $L \equiv L(K)$ such that

$$\int_K |D_x f(\cdot,t,\xi)|(\Omega)\,\mathrm{d}t\mathrm{d}\xi < L. \tag{1.7}$$

Then the functional F is lower semicontinuous in $W^{1,1}(\Omega)$ with respect to the $L^1(\Omega)$ convergence.

We remark that this result is optimal in the sense that, as shown by various examples (see [4] or [15]), if no coercivity nor strict convexity of f is assumed the lower semicontinuity may fail when f is not BV in x. It would be nice to extend this theorem to the case where u is in BV, i.e. to find the lower semicontinuous envelope $\overline{F}$ of the functional F in (1.3). Indeed, in the same spirit of Theorem 1.2 we are able to prove that if $u \in BV(\Omega)$ the absolutely continuous part of $\overline{F}(u)$ is still represented by $\int_\Omega f(x.u.\nabla u)\mathrm{d}x$, where ∇u stands for the absolutely continuous part of the distributional derivative Du (see Theorem 3.5). However, the representation of the singular part of $\overline{F}(u)$ is still an open problem under the quite general assumptions on f made above.

As an example, consider the functional

$$\int_\Omega a(x)|\nabla u(x)|\,\mathrm{d}x,$$

where a is a locally bounded nonnegative BV function. By Theorem 1.2, this functional is L^1-lower semicontinuos on $W^{1,1}$, but even in this case it is not clear which is its lower semicontinuous extension to BV. One could think that a good candidate is the functional

$$\int_\Omega a^*(x)\,\mathrm{d}|Du|(x) \quad u \in BV(\Omega), \tag{1.8}$$

but it is not so. In fact if $\Omega = (0,1)^2, a = \chi_E$, where $E = (0,1) \times (1/2,1)$, the integral in (1.8) is not lower semicontinuous along the sequence $u_n = \chi_{E_n}$, where $E_n = (0,1) \times ((n-1)/(2n),1)$, which converges in $L^1(\Omega)$ to χ_E.

2 Definitions and preliminaries

In this section we recall some preliminary results and basic definitions. For all the material contained in this section the reader may refer to [2,10].

Let E be a measurable subset of $\mathbb{R}^N$. The *density* $D(E;x)$ of E at a point $x \in \mathbb{R}^N$ is defined by

$$D(E;x) = \lim_{\varrho\to 0} \frac{\mathcal{L}^N(E \cap B_\rho(x))}{\omega_N \rho^N},$$

where ω_N is the measure of the unit ball, whenever this limit exists. Hereafter, $B_\rho(x)$ denotes the ball centered at x with radius ρ. The *essential boundary* $\partial^M E$ of E is the Borel set defined as

$$\partial^M E = \mathbb{R}^N \setminus \{x \in \mathbb{R}^N : D(E;x) = 0 \text{ or } D(E;x) = 1\}\,. \tag{2.1}$$

We say that the set E is of *finite perimeter* in an open set Ω if $\mathcal{H}^{N-1}(\partial^M E \cap \Omega) < \infty$. Notice that, by [10, Theorem 4.5.11], this definition is equivalent to the one originally given in [8] and usually adopted in the literature. Notice also that if $\Omega \subset \mathbb{R}^N$ is an open set, the quantity $\mathcal{H}^{N-1}(\partial^M E \cap \Omega)$ agrees with the classical *perimeter of E in* Ω (see [2, Theorem 3.61]).

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u : \Omega \to \mathbb{R}$ be a measurable function. The *upper and lower approximate limits* of u at a point $x \in \Omega$ are defined as

$$u^+(x) = \inf\{t \in \mathbb{R} : D(\{u > t\};x) = 0\}, \quad u^-(x) = \sup\{t \in \mathbb{R} : D(\{u < t\};x) = 0\}, \tag{2.2}$$

respectively. The quantities $u^+(x)$, $u^-(x)$ are well defined (possibly equal to $\pm\infty$) at every $x \in \Omega$, and $u^-(x) \le u^+(x)$. The functions $u^+, u^- : \Omega \to [-\infty,\infty]$ are Borel measurable.

We say that u is *approximately continuous* at a point $x \in \Omega$ if $u^+(x) = u^-(x) \in \mathbb{R}$. In this case, we set $\widetilde{u}(x) = u^+(x) = u^-(x)$ and call $\widetilde{u}(x)$ the *approximate limit* of u at x. Notice that definition (2.2) implies that u is approximately continuous at x with approximate limit $\widetilde{u}(x)$ if and only if

$$D(\{y \in \Omega : |u(y) - \widetilde{u}(x)| > \varepsilon\}; x) = 0 \quad \text{for every } \varepsilon > 0\,. \tag{2.3}$$

The set of all points in Ω where u is approximately continuous is a Borel set which will be denoted by C_u and called the set of *approximate continuity* of u. The set $S_u = \Omega \setminus C_u$ will be referred to as the set of *approximate discontinuity* of u.

As a simple consequence of the above definitions we have,

$$\partial^M\{u > t\} \subset \{u^- \le t \le u^+\} \quad \text{for every } t \in \mathbb{R}. \tag{2.4}$$

In particular,

$$C_u \cap \partial^M\{u > t\} \subset \{\widetilde{u} = t\} \quad \text{for every } t \in \mathbb{R}. \tag{2.5}$$

Finally, by u^* we denote the *precise representative* of u which is defined by

$$u^*(x) = \frac{u^+(x) + u^-(x)}{2}$$

if $u^+(x), u^-(x) \in \mathbb{R}$, $u^*(x) = 0$ otherwise.

A locally integrable function u is said to be *approximately differentiable* at a point $x \in C_u$ if there exists $\nabla u(x) \in \mathbb{R}^N$ such that

$$\lim_{\rho \to 0} \frac{1}{\rho^{N+1}} \int\limits_{B_\rho(x)} |u(y) - \widetilde{u}(x) - \langle \nabla u(x), y - x\rangle|\, dy = 0. \tag{2.6}$$

Here, $\langle \cdot, \cdot \rangle$ stands for scalar product in $\mathbb{R}^N$. The vector $\nabla u(x)$ is called the *approximate differential* of u at x. The set of all points in C_u where u is approximately differentiable is denoted by $\mathcal{D}_u$ and is called the set of *approximate differentiability* of u. It can be easily verified that $\mathcal{D}_u$ is a Borel set and that $\nabla u : \mathcal{D}_u \to \mathbb{R}^N$ is a Borel function.

A function $u \in L^1(\Omega)$ is said to be of *bounded variation* if its distributional gradient Du is an $\mathbb{R}^N$-valued Radon measure in Ω and the total variation $|Du|$ of Du is finite in Ω. The space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$, while the notation $BV_{\text{loc}}(\Omega)$ will be reserved for the space of those functions $u \in L^1_{\text{loc}}(\Omega)$ such that $u \in BV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

Let $u \in BV(\Omega)$. Then it can be proved that

$$\lim_{\rho \to 0} \fint\limits_{B_\rho(x)} |u(y) - \widetilde{u}(x)|\, dy = 0 \quad \text{for } \mathcal{H}^{N-1}\text{–a.e. } x \in C_u$$

and that u is approximately differentiable for $\mathcal{L}^N$-a.e. x. Moreover, the functions u^- and u^+ are finite $\mathcal{H}^{N-1}$-a.e. and for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ there exists a unit vector $\nu_u(x)$ such that

$$\lim_{\rho \to 0} \fint\limits_{B^+_\rho(x;\nu_u(x))} |u(y) - u^+(x)|\, dy = 0, \quad \lim_{\rho \to 0} \fint\limits_{B^-_\rho(x;\nu_u(x))} |u(y) - u^-(x)|\, dy = 0, \tag{2.7}$$

where $B^+_\rho(x; \nu_u(x)) = \{y \in B_\rho(x) : \langle y - x, \nu_u(x)\rangle > 0\}$, and $B^-_\varrho(x; \nu_u(x))$ is defined analogously. The set of all points in S_u where the equalities in (2.7) are satisfied is called the *jump set* of u and is denoted by J_u.

If u is a BV function, we denote by $D^a u$ the absolutely continuous part of Du with respect to Lebesgue measure. The singular part, denoted by $D^s u$, is split into two more parts, the *jump part* $D^j u$ and the *Cantor part* $D^c u$, defined by

$$D^j u = D^s u \llcorner J_u, \quad D^c u = D^s u - D^j u.$$

Finally, we denote by $\widetilde{D}u$ the *diffuse part* of Du, defined by

$$\widetilde{D}u = D^a u + D^c u.$$

If $u \in BV(\Omega)$, then for a.e. $t \in \mathrm{I\!R}$ the set $\{x \in \Omega : u(x) > t\}$ is of finite perimeter in Ω. Moreover, the following version of the *coarea formula* holds (see [2, Theorem 3.40 and (3.63)]).

Theorem 2.1 **(**Coarea formula) *Let* Ω *be an open subset of* $\mathrm{I\!R}^N$ *and let* $u \in BV(\Omega)$. *Assume that* $g : \Omega \to [0, +\infty]$ *is a Borel function. Then*

$$\int_\Omega g \, \mathrm{d}|Du| = \int_{-\infty}^{+\infty} \mathrm{d}t \int_{\partial^M \{u>t\} \cap \Omega} g \, \mathrm{d}\mathcal{H}^{N-1}. \tag{2.8}$$

An alternative version of formula (2.8) states that

$$\int_\Omega g \, \mathrm{d}|Du| = \int_{-\infty}^{+\infty} \mathrm{d}t \int_{\{u^- \le t \le u^+\}} g \, \mathrm{d}\mathcal{H}^{N-1} \tag{2.9}$$

(see [10, Theorem 4.5.9]). Making use of (2.8) and (2.9) with $g \equiv 1$ and of (2.4) yields

$$\mathcal{H}^{N-1}\Big(\{u^- \le t \le u^+\} \setminus \big(\partial^M \{u > t\} \cap \Omega\big)\Big) = 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathrm{I\!R}. \tag{2.10}$$

The following lemma contains some useful properties of the characteristic functions of the level sets of a BV function u.

Lemma 2.2 *Let* Ω *be an open subset of* $\mathrm{I\!R}^N$ *and let* $u : \Omega \to \mathrm{I\!R}$ *a measurable function. Then, for all* $t \in \mathrm{I\!R}$ *and* $x \in \Omega$

$$u^-(x) > t \implies \chi^*_{\{u>t\}}(x) = 1, \quad u^+(x) < t \implies \chi^*_{\{u>t\}}(x) = 0, \tag{2.11}$$

Moreover, if $u \in BV(\Omega)$, *for* $\mathcal{L}^1$*-a.e.* $t \in \mathrm{I\!R}$ *there exists a Borel set* $N_t \subset \Omega$, *with* $\mathcal{H}^{N-1}(N_t) = 0$, *such that for any* $x \in \Omega \setminus N_t$ *the following relations hold*

$$u^-(x) > t \iff \chi^*_{\{u>t\}}(x) = 1, \quad u^+(x) < t \iff \chi^*_{\{u>t\}}(x) = 0, \tag{2.12}$$

$$u^-(x) \le t \le u^+(x) \iff \chi^*_{\{u>t\}}(x) = \frac{1}{2}. \tag{2.13}$$

Proof Let us fix $x \in \Omega$ and assume that $u^-(x) > t$. Then, by the definition (2.2), it follows that $D\big(\{u < s\}; x\big) = 0$ for all $t < s < u^-(x)$. In particular, we have that $D\big(\{u > t\}; x\big) = 1$, i.e. $D\big(\{\chi_{\{u>t\}} = 1\}; x\big) = 1$ which, in turn, is equivalent to $\chi^-_{\{u>t\}}(x) = 1$. From this equality we immediately get that $\chi^*_{\{u>t\}}(x) = 1$, since $\chi^-_{\{u>t\}}(x) \le \chi^+_{\{u>t\}}(x) \le 1$ for all $x \in \Omega$. The other implication in (2.11) is proved in the same way.

To prove (2.12) and (2.13), notice that there exists an $\mathcal{L}^1$-null set $T \subset \mathbb{R}$ such that if $t \notin T$ the level set $\{u > t\}$ is of finite perimeter in Ω and (2.10) holds. Let us fix $t \in \mathbb{R} \setminus T$. Since $\{u > t\}$ is a set of finite perimeter, denoting by $\{u > t\}^{1/2}$ the set of points $x \in \mathbb{R}^N$ such that $D\big(\{u > t\}; x\big) = 1/2$, we have (see [2, Theorem 3.61]) $\mathcal{H}^{N-1}\big(\partial^M\{u > t\} \cap \Omega \setminus \{u > t\}^{1/2}\big) = 0$. Therefore, from this equation and from (2.10), setting

$$N_t = \{u^- \leq t \leq u^+\} \setminus \{u > t\}^{1/2}$$

we have that $\mathcal{H}^{N-1}(N_t) = 0$.

Let x be a point in $\Omega \setminus N_t$, such that $\chi^*_{\{u>t\}}(x) = 1$. From this equality we have that $\chi^-_{\{u>t\}}(x) = 1$, which is equivalent to the equality $D\big(\{u > t\}; x\big) = 1$, hence $D\big(\{u < t\}; x\big) = 0$ and, by (2.2), this last equality yields $u^-(x) \geq t$. However, if $u^-(x)$ were equal to t, then x would trivially satisfy the inequality $u^-(x) \leq t \leq u^+(x)$ and this is impossible since $D\big(\{u > t\}; x\big) = 1$, hence $x \notin \{u > t\}^{1/2}$, and by assumption $x \notin N_t$. Therefore $u^-(x) > t$ and by (2.11) we obtain the first equivalence in (2.12). The second equivalence is proved similarly.

If $u^-(x) \leq t \leq u^+(x)$ and $x \notin N_t$, then necessarily $x \in \{u > t\}^{1/2}$, hence we easily get that $\chi^*_{\{u>t\}}(x) = 1/2$. The opposite implication follows trivially from (2.12). □

Next result is contained in [2, Lemma 2.35].

Lemma 2.3 *Let μ be a positive Radon measure in an open set $\Omega \subset \mathbb{R}^N$ and let $\psi_j : \Omega \to [0, \infty]$, $j \in \mathbb{N}$, be Borel functions. Then*

$$\int_\Omega \sup_j \psi_j \, \mathrm{d}\mu = \sup \left\{ \sum_{j \in J} \int_{A_j} \psi_j \, \mathrm{d}\mu \right\},$$

where the supremum ranges among all finite sets $J \subset \mathbb{N}$ and all families $\{A_j\}_{j\in\mathbb{N}}$ of pairwise disjoint open sets with compact closure in Ω.

The following lemma is a classical approximation result due to De Giorgi (see [9]).

Lemma 2.4 *Let f be a locally bounded Borel function from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $[0, +\infty)$, satisfying (1.4). Then, there exists a sequence $\{\alpha_k\} \subset C_0^\infty(\mathbb{R}^N)$, with $\alpha_k \geq 0$ and $\int_{\mathbb{R}^N} \alpha_k(\xi)\, d\xi = 1$ such that, if we set for $(x, t) \in \Omega \times \mathbb{R}, i = 0, 1, \ldots, N$,*

$$a_{0,k}(x, t) = \int_{\mathbb{R}^N} f(x, t, \xi)\, ((N + 1)\alpha_k(\xi) + \langle \nabla\alpha_k(\xi), \xi\rangle) \mathrm{d}\xi,$$

$$a_{i,k}(x, t) = -\int_{\mathbb{R}^N} f(x, t, \xi) \frac{\partial}{\partial \xi_i} \alpha_k(\xi) \mathrm{d}\xi,$$

and, for $(x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$,

$$g_k(x, t, \xi) = a_{0,k}(x, t) + \sum_{i=1}^N a_{i,k}(x, t)\xi_i,$$

then, for all $(x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, we have

$$f(x, t, \xi) = \sup_k \max\{g_k(x, t, \xi), 0\}.$$

Remark 2.5 We remark that if for all $(x,\xi) \in \Omega \times \mathbb{R}^N$ the function $f(x,\cdot,\xi)$ is continuous, then for every $x \in \Omega$ the coefficients $a_{i,k}(x,\cdot)$ are continuous functions. If $f(\cdot,t,\xi)$ is a BV function for every $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and (1.7) holds, then for every $k \in \mathbb{N}, i = 0,\dots,N$, and $t \in \mathbb{R}$ the coefficients $a_{i,k}(\cdot,t)$ given by the previous lemma are BV functions. Moreover, it can be easily checked that for every compact $H \subset \mathbb{R}$, there exists $L_k \equiv L_k(H)$ such that

$$\int_H |D_x a_k(\cdot,t)|(\Omega)\,\mathrm{d}t \le L_k, \tag{2.14}$$

where $a_k = (a_{1,k},\dots,a_{N,k})$.

3 Proofs

We start this section with a simple technical lemma.

Lemma 3.1 *Let $b(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally bounded Borel function. Then, the function $b^*(x,t)$ is also a locally bounded Borel function in $\Omega \times \mathbb{R}$.*

Proof Let us prove that the function defined by

$$b^+(x,t) := \inf\{s \in \mathbb{R} : \ D(\{y : b(y,t) > s\};x) = 0\} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}$$

is a Borel function. Then the result will follow at once, since $b^*(x,t) = (b^+(x,t) + b^-(x,t))/2$, where b^-, defined similarly to b^+, is also a Borel function.

First, notice that for all $a \in \mathbb{R}$

$$\{(x,t) : \ b^+(x,t) < a\} = \bigcup_{s_i < a, s_i \in \mathbb{Q}} \{(x,t) : \ D(\{y : b(y,t) > s_i\};x) = 0\}.$$

Thus, it is enough to show that for every $s \in \mathbb{R}$ the set $\{(x,t) : \ D(\{y : b(y,t) > s\};x) = 0\}$ is a Borel set or, equivalently, that the set

$$\left\{(x,t) : \ \limsup_{\rho\to 0} \frac{1}{\rho^N}\mathcal{L}^N(\{y : b(y,t) > s\} \cap B_\rho(x)) = 0\right\}$$

is a Borel set. To this aim, it is enough to prove that the function

$$(x,t) \in \Omega \times \mathbb{R} \to \inf_{i\in\mathbb{N}} \sup_{\rho\in\mathbb{Q}, 0<\rho<\frac{1}{i}} \frac{1}{\rho^N}\mathcal{L}^N(\{y : b(y,t) > s\} \cap B_\rho(x))$$

is a Borel function, and this follows at once from the fact that for any Borel set $A \subset \mathbb{R}^N \times \mathbb{R}$ the function

$$(x,t) \in \mathbb{R}^N \times \mathbb{R} \to \mathcal{L}^N(A_t \cap B_\rho(x)),$$

where $A_t = \{y : (y,t) \in A\}$, is a Borel function. In order to prove this last property, let us consider the family of subsets of $\mathbb{R}^N \times \mathbb{R}$ defined by

$$\mathcal{F} = \{A \text{ is a Borel subset of } \mathbb{R}^N \times \mathbb{R} : \ (x,t) \to \mathcal{L}^N(A_t \cap B_\rho(x)) \text{ is a Borel function}\}.$$

The following properties of $\mathcal{F}$ are easily cheched: (i) if A_i is an increasing sequence of sets in $\mathcal{F}$, then $\cup_i A_i$ belongs to $\mathcal{F}$; (ii) if A_1, A_2 and $A_1 \cup A_2$ belong to $\mathcal{F}$, then

$A_1 \cap A_2 \in \mathcal{F}$; (iii) if $A \in \mathcal{F}$, then $\mathbb{R}^N \setminus A \in \mathcal{F}$. From all these properties and from the fact that $\mathcal{F}$ contains any product of a Borel subset of $\mathbb{R}^N$ and of a Borel subset of $\mathbb{R}$, using [2, Remark 1.9], we get that $\mathcal{F}$ coincides with the family of Borel sets in $\mathbb{R}^N \times \mathbb{R}$. Hence, the result follows. □

Remark 3.2 We claim that, if u is a measurable, locally bounded function from Ω to $\mathbb{R}$, then the function $(x,t) : \Omega \times \mathbb{R} \to \chi^*_{\Omega_{u,t}}(x)$ is Borel. In fact notice that since u^* is a Borel function, the function $(x,t) : \Omega \times \mathbb{R} \to \chi_{\Omega_{u^*,t}}(x)$ is Borel too. Hence, Lemma 3.1 yields that $(x,t) \to \chi^*_{\Omega_{u^*,t}}(x)$ is Borel. Therefore, since $\chi_{\Omega_{u^*,t}}(x) = \chi_{\Omega_{u,t}}(x)$ for $\mathcal{L}^N$-a.e. x and for all t, the claim follows from the fact that, for all $x \in \Omega$ and all $t \in \mathbb{R}$, $\chi^*_{\Omega_{u,t}}(x) = \chi^*_{\Omega_{u^*,t}}(x)$.

Proof of Theorem 1.1 Step 1 Let us fix a test function $\phi \in C^1_0(\Omega)$ and let $\Omega' \subset\subset \Omega$ be an open set such that supp$\phi \subset \Omega'$. Denote by $\varrho_\varepsilon(x) = \varepsilon^{-N}\varrho(x/\varepsilon)$, where $\varepsilon <$dist$(\Omega', \partial\Omega)$, a standard radially symmetric mollifier and define

$$b_\varepsilon(x,t) := \int_\Omega \varrho_\varepsilon(x-y)\, b(y,t)\,\mathrm{d}y$$

for all $x \in \Omega'$ and $t \in \mathbb{R}$.

Given a function $u \in BV(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ we define

$$v_\varepsilon(x) := \int_0^{u(x)} b_\varepsilon(x,t)\mathrm{d}t$$

and since the function $(x,t) \to \int_0^t b_\varepsilon(x, s)\mathrm{d}s$ is locally Lipschitz in $\Omega \times \mathbb{R}$, using a general chain rule formula due to Ambrosio and Dal Maso (see [1] and [2, Theorem 3.101]), we have that $v_\varepsilon \in BV(\Omega')$ and

$$\begin{aligned}\int_\Omega \nabla\phi(x) v_\varepsilon(x)\mathrm{d}x = &-\int_\Omega \phi(x)\,\mathrm{d}x \int_0^{u(x)} \nabla_x b_\varepsilon(x,t)\mathrm{d}t \\ &-\int_\Omega \phi(x) b_\varepsilon(x,\widetilde{u}(x))\nabla u(x)\mathrm{d}x \\ &-\int_\Omega \phi(x) b_\varepsilon(x,\widetilde{u}(x))\mathrm{d}D^c u \\ &-\int_{J_u} \phi(x)\left[\int_{u^-(x)}^{u^+(x)} b_\varepsilon(x,t)\mathrm{d}t\right] \nu_u(x)\mathrm{d}\mathcal{H}^{N-1}(x). \qquad (3.1)\end{aligned}$$

Let us now prove that

$$\lim_{\varepsilon\to 0^+} \int_\Omega \nabla\phi(x)\,\mathrm{d}x \int_0^{u(x)} b_\varepsilon(x,t)\,\mathrm{d}t = \int_\Omega \nabla\phi(x)\,\mathrm{d}x \int_0^{u(x)} b(x,t)\,\mathrm{d}t. \qquad (3.2)$$

To this aim, it is enough to observe that

$$\left|\int_{\Omega} \nabla\phi \mathrm{d}x \int_{0}^{u(x)} b_{\varepsilon}(x,t)\mathrm{d}t - \int_{\Omega} \nabla\phi \mathrm{d}x \int_{0}^{u(x)} b(x,t)\mathrm{d}t\right|$$
$$\leq \|\nabla\phi\|_{\infty} \int_{-M}^{M} \mathrm{d}t \int_{\Omega'} \chi_{\Omega_{u,t}}(x)|b_{\varepsilon}(x,t) - b(x,t)|\mathrm{d}x,$$

where $\Omega_{u,t} = \{x \in \Omega : t$ belongs to the segment of endpoints 0 and $u(x)\}$ and M is a positive number such that $\|u\|_{\infty} < M$. Then, recalling that for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ the functions $b_{\varepsilon}(\cdot,t)$ converge in $L^1(\Omega')$ to $b(\cdot,t)$, (3.2) follows from Lebesgue's dominated convergence theorem.

Step 2 We shall prove separately the convergence of the diffuse and jump parts, i.e.

$$\lim_{\varepsilon\to 0^+} \int_{\Omega} \phi(x) b_{\varepsilon}(x,\widetilde{u}(x))\, \mathrm{d}\widetilde{D}u(x) = \int_{\Omega} \phi(x) b^*(x,\widetilde{u}(x))\, \mathrm{d}\widetilde{D}u(x) \tag{3.3}$$

and

$$\lim_{\varepsilon\to 0^+} \int_{J_u} \phi(x) \left[\int_{u_-(x)}^{u_+(x)} b_{\varepsilon}(x,t)\mathrm{d}t\right] \nu_u(x)\mathrm{d}\mathcal{H}^{N-1} = \int_{J_u} \phi(x) \left[\int_{u_-(x)}^{u_+(x)} b^*(x,t)\mathrm{d}t\right] \nu_u(x)\mathrm{d}\mathcal{H}^{N-1}. \tag{3.4}$$

Using the coarea formula (2.9), we get

$$\begin{aligned}\int_{\Omega} \phi(x) b_{\varepsilon}(x,\widetilde{u}(x))\, \mathrm{d}\widetilde{D}u &= \int_{C_u} \phi(x) b_{\varepsilon}(x,\widetilde{u}(x)) \frac{\widetilde{D}u}{|Du|}(x)\, \mathrm{d}|Du| \\ &= \int_{-\infty}^{+\infty} \mathrm{d}t \int_{\{u^- \leq t \leq u^+\}} \phi(x) b_{\varepsilon}(x,\widetilde{u}(x)) \chi_{C_u}(x) \frac{\widetilde{D}u}{|Du|}(x)\, \mathrm{d}\mathcal{H}^{N-1} \\ &= \int_{-\infty}^{+\infty} \mathrm{d}t \int_{\{\widetilde{u}=t\}\cap C_u} \phi(x) b_{\varepsilon}(x,t) \frac{\widetilde{D}u}{|Du|}(x)\, \mathrm{d}\mathcal{H}^{N-1}. \end{aligned} \tag{3.5}$$

Now, recall that for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ the function $b(\cdot,t) \in BV(\Omega)$, hence (see Propositions 3.64(b) and 3.69(b) in [2])

$$b_{\varepsilon}(x,t) \to b^*(x,t) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega \tag{3.6}$$

as $\varepsilon \to 0$. Therefore, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, we have

$$\lim_{\varepsilon\to 0} \int_{\{\widetilde{u}=t\}\cap C_u} \phi(x) b_{\varepsilon}(x,t) \frac{\widetilde{D}u}{|Du|}(x)\, \mathrm{d}\mathcal{H}^{N-1} = \int_{\{\widetilde{u}=t\}\cap C_u} \phi(x) b^*(x,t) \frac{\widetilde{D}u}{|Du|}(x)\, \mathrm{d}\mathcal{H}^{N-1}.$$

From this equation, using the local boundedness of b^* and the fact that, by the coarea formula (2.9),

$$\int_{-\infty}^{+\infty} \mathcal{H}^{N-1}\left(\{\widetilde{u}=t\} \cap C_u\right) \mathrm{d}t = |Du|(C_u) < \infty,$$

we can pass to the limit in (3.5) and by the Lebesgue's dominated convergence theorem we get

$$\lim_{\varepsilon\to 0} \int_{\Omega} \phi(x) b_\varepsilon(x, \widetilde{u}(x))\, \mathrm{d}\widetilde{D}u = \int_{-\infty}^{+\infty} \mathrm{d}t \int_{\{\widetilde{u}=t\}\cap C_u} \phi(x) b^*(x,t) \frac{\widetilde{D}u}{|Du|}(x)\, \mathrm{d}\mathcal{H}^{N-1}.$$

From this equation, using the coarea formula (2.9) again, we immediately get (3.3). Finally, using Fubini's theorem, we estimate

$$\begin{aligned}
&\left| \int_{J_u} \phi(x) \nu_u(x)\, \mathrm{d}\mathcal{H}^{N-1} \int_{u^-(x)}^{u^+(x)} b_\varepsilon(x,t)\, \mathrm{d}t - \int_{J_u} \phi(x) \nu_u(x)\, \mathrm{d}\mathcal{H}^{N-1} \int_{u^-(x)}^{u^+(x)} b^*(x,t)\, \mathrm{d}t \right| \\
&\le \|\phi\|_\infty \int_{J_u\cap\Omega'} \mathrm{d}\mathcal{H}^{N-1} \int_{u^-(x)}^{u^+(x)} |b_\varepsilon(x,t) - b^*(x,t)|\, \mathrm{d}t \\
&\le \|\phi\|_\infty \int_{J_u\cap\{x\in\Omega' : u^+(x)-u^-(x)<1/h\}} \mathrm{d}\mathcal{H}^{N-1} \int_{u^-(x)}^{u^+(x)} |b_\varepsilon(x,t) - b^*(x,t)|\, \mathrm{d}t \\
&\quad + \|\phi\|_\infty \int_{J_u\cap\{x\in\Omega' : u^+(x)-u^-(x)\ge 1/h\}} \mathrm{d}\mathcal{H}^{N-1} \int_{u^-(x)}^{u^+(x)} |b_\varepsilon(x,t) - b^*(x,t)|\, \mathrm{d}t.
\end{aligned} \tag{3.7}$$

Notice that for all $\varepsilon > 0$ and $h \in \mathbb{N}$

$$\begin{aligned}
&\int_{J_u\cap\{x\in\Omega' : u^+(x)-u^-(x)<1/h\}} \mathrm{d}\mathcal{H}^{N-1} \int_{u^-(x)}^{u^+(x)} |b_\varepsilon(x,t) - b^*(x,t)|\, \mathrm{d}t \\
&\le 2\|b\|_{L^\infty(\Omega'\times(-M,M))} \int_{J_u\cap\{x\in\Omega' : u^+(x)-u^-(x)<1/h\}} |u^+(x) - u^-(x)|\, \mathrm{d}\mathcal{H}^{N-1}.
\end{aligned} \tag{3.8}$$

On the other hand,

$$\begin{aligned}
&\int_{J_u\cap\{x\in\Omega' : u^+(x)-u^-(x)\ge 1/h\}} \mathrm{d}\mathcal{H}^{N-1} \int_{u^-(x)}^{u^+(x)} |b_\varepsilon(x,t) - b^*(x,t)|\, \mathrm{d}t \\
&= \int_{-M}^{M} \mathrm{d}t \int_{J_u\cap\{x\in\Omega' : u^+(x)-u^-(x)\ge 1/h\}} \chi_{[u^-(x),u^+(x)]}(t) |b_\varepsilon(x,t) - b^*(x,t)|\, \mathrm{d}\mathcal{H}^{N-1}
\end{aligned}$$

and this last integral is infinitesimal as $\varepsilon \to 0$, since for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have that $b_\varepsilon(x,t) \to b^*(x,t)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$, b^* and b_ε are bounded and, for any h, $J_u \cap \{x \in \Omega : u^+(x) - u^-(x) \geq 1/h\}$ is a set of finite $\mathcal{H}^{N-1}$ measure. Therefore, from (3.8) and (3.7), letting first ε tend to zero and then h tend to ∞, we immediately obtain (3.4).

Step 3 Notice that since $b(\cdot,t) \in BV(\Omega)$ for $\mathcal{L}^1$-a.e. t we have that for every $x \in \Omega'$

$$\nabla_x b_\varepsilon(x,t) = \int_\Omega \varrho_\varepsilon(x-y)\,\mathrm{d}D_y b(y,t).$$

Thus, we have by Fubini's theorem,

$$\begin{aligned}
\int_\Omega \phi(x)\,dx \int_0^{u(x)} \nabla_x b_\varepsilon(x,t)\,\mathrm{d}t &= \int_{-M}^{M} \mathrm{d}t \int_{\Omega_{u,t}} \operatorname{sgn}(t)\phi(x)\nabla_x b_\varepsilon(x,t)\,\mathrm{d}x \\
&= \int_{-M}^{M} \mathrm{d}t \int_{\Omega_{u,t}} \operatorname{sgn}(t)\phi(x)\,\mathrm{d}x \int_\Omega \varrho_\varepsilon(x-y)\,\mathrm{d}D_y b(y,t) \\
&= \int_{-M}^{M} \operatorname{sgn}(t)\,\mathrm{d}t \int_\Omega \mathrm{d}D_y b(y,t) \int_\Omega \varrho_\varepsilon(x-y)\phi(x)\chi_{\Omega_{u,t}}(x)\,\mathrm{d}x \\
&= \int_{-M}^{M} \operatorname{sgn}(t)\,\mathrm{d}t \int_\Omega \varrho_\varepsilon * (\phi\chi_{\Omega_{u,t}})(y)\,\mathrm{d}D_y b(y,t).
\end{aligned} \tag{3.9}$$

For $\mathcal{L}^1$-a.e. t the function $\chi_{\Omega_{u,t}}$, being the characteristic function of a set of finite perimeter, is in $BV(\Omega)$. Therefore for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ (hence, for $|D_x b(\cdot,t)|$-a.e. $x \in \Omega$) we have that

$$\lim_{\varepsilon\to 0^+} \varrho_\varepsilon * (\phi\chi_{\Omega_{u,t}})(x) = \phi(x)\chi^*_{\Omega_{u,t}}(x)$$

and thus

$$\lim_{\varepsilon\to 0^+} \int_\Omega \varrho_\varepsilon * (\phi\chi_{\Omega_{u,t}})(x)\,\mathrm{d}D_x b(x,t) = \int_\Omega \phi(x)\chi^*_{\Omega_{u,t}}(x)\,\mathrm{d}D_x b(x,t).$$

From this equation, using the assumption (ii) and the Lebesgue's dominated convergence theorem, we can pass to the limit in (3.9), thus getting that

$$\lim_{\varepsilon\to 0^+} \int_\Omega \phi(x)\,\mathrm{d}x \int_0^{u(x)} \nabla_x b_\varepsilon(x,t)\,\mathrm{d}t = \int_{-M}^{M} \mathrm{d}t \int_\Omega \operatorname{sgn}(t)\phi(x)\chi^*_{\Omega_{u,t}}(x)\,\mathrm{d}D_x b(x,t).$$

Then, the assertion follows at once from the last equality, (3.2), (3.3), (3.4) and from equation (3.1). □

Remark 3.3 Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, such that $f(0) = 0$. Setting $b(t) = f'(t)$ and $v(x) = f(u(x))$, from (1.2) we get that

$$Dv = f'(\widetilde{u}(x))\nabla u\mathcal{L}^N + f'(\widetilde{u}(x))D^c u + \big(f(u^+(x)) - f(u^-(x))\big)\nu_u\mathcal{H}^{N-1} \llcorner J_u\,,$$

which agrees with the 'classical' chain rule formula for BV functions (see [2, Theorem 3.96]). Assume now that $b(x,t) \equiv b(x)$ is a bounded $BV(\Omega)$ function and that $u \in BV(\Omega) \cap L^\infty(\Omega)$ and use (1.2) to deduce a formula for the derivative of bu. To this aim, notice that Lemma 2.2 yields that, for $\mathcal{L}^1$-a.e. $t > 0$, $\chi^*_{\Omega_{u,t}}(x) = 1$ if $u^-(x) > t$, $\chi^*_{\Omega_{u,t}}(x) = 1/2$ if $u^-(x) \le t \le u^+(x)$ and $\chi^*_{\Omega_{u,t}}(x) = 0$ if $t > u^+(x)$, and similar formulas hold if $t < 0$, therefore

$$\int_{-\infty}^{+\infty} \mathrm{sgn}(t)\chi^*_{\Omega_{u,t}}(x)\,\mathrm{d}t = \frac{u^+(x) + u^-(x)}{2} \quad \text{and} \quad \int_{u^-(x)}^{u^+(x)} b^*(x)\,\mathrm{d}t = b^*(x)(u^+(x) - u^-(x)).$$

From these equalities and from (1.2), we then get easily that the diffuse part $\widetilde{D}(bu)$ of the distributional derivative of bu is given by

$$\widetilde{D}(bu) = \widetilde{u}(x)\widetilde{D}b + \widetilde{b}(x)\widetilde{D}u\,,$$

while the representation formula for the jump part of $D(bu)$ splits in three parts:

$$D^j(bu) = \widetilde{u}(x)D^j b \llcorner (J_b \setminus J_u) + \widetilde{b}(x)D^j u \llcorner (J_u \setminus J_b) + D^j(bu)\llcorner (J_b \cap J_u)\,,$$

where

$$D^j(bu)\llcorner(J_b \cap J_u) = \begin{cases} \big(b^+(x)u^+(x) - b^-(x)u^-(x)\big)\nu_b(x)\mathcal{H}^{N-1} & \text{if } \nu_b(x) = \nu_u(x), \\ \\ \big(b^+(x)u^-(x) - b^-(x)u^+(x)\big)\nu_b(x)\mathcal{H}^{N-1} & \text{if } \nu_b(x) = -\nu_u(x), \end{cases}$$

(see [2, Example 3.97]). Therefore, we may conclude that

$$D(bu) = u^* Db + b^* Du.$$

By a slight modification of the proof of Theorem 1.1 we can deduce the following chain rule for Sobolev functions. To this aim, let us introduce the space

$$M^1(\mathrm{div};\Omega) = \{v \in L^1(\Omega;\mathbb{R}^N) : \ \mathrm{div} v \ \text{ is a Radon measure in } \Omega\}\,.$$

Theorem 3.4 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $b : \Omega \times \mathbb{R} \to \mathbb{R}^N$ be a locally bounded Borel function. Assume that*

(i) *for $\mathcal{L}^N$-a.e. $x \in \Omega$ the function $b(x,\cdot)$ is continuous in $\mathbb{R}$;*
(ii) *for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ the function $b(\cdot,t) \in M^1(\mathrm{div};\Omega)$;*
(iii) *for any compact set $H \subset \mathbb{R}$,*

$$\int_H |\mathrm{div}_x b(\cdot,t)|(\Omega)\,\mathrm{d}t < +\infty.$$

Then, for every $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, the function $v : \Omega \to \mathbb{R}^N$, defined by

$$v(x) := \int_0^{u(x)} b(x,t)\,\mathrm{d}t,$$

belongs to $M^1(\mathrm{div};\Omega)$ *and for any* $\phi \in C_0^1(\Omega)$ *we have*

$$\int_\Omega \langle \nabla\phi(x), v(x)\rangle \mathrm{d}x = -\int_{-\infty}^{+\infty} \mathrm{d}t \int_\Omega \mathrm{sgn}(t)\chi^*_{\Omega_{u,t}}\phi(x)\mathrm{d}\,\mathrm{div}_x b(x,t)$$
$$-\int_\Omega \phi(x)\langle b(x,u(x)), \nabla u(x)\rangle \mathrm{d}x.$$

Proof Let us fix a function $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and a test function $\phi \in C_0^1(\Omega)$ and define b_ε and v_ε as in the proof of Theorem 1.1. Since b_ε is locally Lipschitz in $\Omega \times \mathbb{R}$, we have that $v_\varepsilon \in W^{1,1}(\Omega;\mathbb{R}^N)$ and

$$\int_\Omega \langle \nabla\phi(x), v_\varepsilon(x)\rangle \mathrm{d}x = -\int_\Omega \phi(x)\mathrm{d}x \int_0^{u(x)} \mathrm{div}_x b_\varepsilon(x,t)\mathrm{d}t$$
$$-\int_\Omega \phi(x)\langle b_\varepsilon(x,u(x)), \nabla u(x)\rangle \mathrm{d}x. \tag{3.10}$$

The convergence of the integral on the left hand side is proved exactly as in the proof of Theorem 1.1. Moreover, the convergence of the last integral on the right hand side follows immediately by observing that for $\mathcal{L}^N$-a.e. $x \in \Omega$

$$\lim_{\varepsilon\to 0^+} b_\varepsilon(x,t) = b(x,t) \quad \text{for all } t \in \mathbb{R}.$$

In fact, this equality follows from the assumption (i), using Scorza–Dragoni's lemma with the same simple argument used below to prove (3.12).

Since

$$\mathrm{div}_x b_\varepsilon(x,t) = \int_\Omega \varrho_\varepsilon(x-y)\,\mathrm{d}\,\mathrm{div}_y b(y,t)\,,$$

arguing as in (3.9), we get that

$$\int_\Omega \phi(x)\,\mathrm{d}x \int_0^{u(x)} \mathrm{div}_x\, b_\varepsilon(x,t)\,\mathrm{d}t = \int_{-M}^{M} \mathrm{sgn}(t)\,\mathrm{d}t \int_\Omega \varrho_\varepsilon * (\phi\chi_{\Omega_{u,t}})(y)\,\mathrm{d}\,\mathrm{div}_y b(y,t), \tag{3.11}$$

where M is a positive number such that $\|u\|_\infty < M$. Since, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, $\mathrm{div}_x b(\cdot,t)$ is a Radon measure, from Proposition 3.1 of [3] we get that, for every Borel subset A of Ω with $\mathcal{H}^{N-1}(A) = 0$, the total variation $|\mathrm{div}_x b(\cdot,t)|(A)$ is zero. On the other hand, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, the function $\chi_{\Omega_{u,t}}$ is the characteristic function of a set of finite perimeter and thus is in $BV(\Omega)$. Therefore, for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$, (hence, for $|\mathrm{div}_x b(\cdot,t)|$-a.e. $x \in \Omega$) we have that

$$\lim_{\varepsilon\to 0^+} \varrho_\varepsilon * (\phi\chi_{\Omega_{u,t}})(x) = \phi(x)\chi^*_{\Omega_{u,t}}(x)$$

and thus

$$\lim_{\varepsilon\to 0^+} \int_\Omega \varrho_\varepsilon * (\phi\chi_{\Omega_{u,t}})(x)\, d\,\mathrm{div}_x b(x,t) = \int_\Omega \phi(x)\chi^*_{\Omega_{u,t}}(x)\,\mathrm{d}\,\mathrm{div}_x b(x,t)\,.$$

Therefore using the assumption (iii) and the Lebesgue's dominated convergence theorem, we can pass to the limit in (3.11), and we obtain

$$\lim_{\varepsilon\to 0^+}\int_\Omega \phi(x)\,\mathrm{d}x\int_0^{u(x)}\mathrm{div}_x b_\varepsilon(x,t)\,\mathrm{d}t=\int_{-M}^{M}\mathrm{d}t\int_\Omega \mathrm{sgn}(t)\phi(x)\chi^*_{\Omega_{u,t}}(x)\,\mathrm{d}\,\mathrm{div}_x b(x,t).$$

Then, the assertion follows from this equality and from (3.10). □

Proof of Theorem 1.2. Step 1. Let (u_n) be a sequence in $W^{1,1}(\Omega)$ converging in $L^1(\Omega)$ to $u\in W^{1,1}(\Omega)$. We may assume, without loss of generality, that $u_n(x)\to u(x)$ for $\mathcal{L}^N$-a.e. $x\in\Omega$. Let us introduce the Borel set $G=\{x\in\Omega:\widetilde{u}_n(x)\to\widetilde{u}(x)\}$ and fix an open set $\Omega'\subset\subset\Omega$ and a function $\eta\in C^1_0(\mathbb{R})$, with $0\le\eta(t)\le 1$.

Denoting by g_k the sequence of functions provided by Lemma 2.4, we fix k. Notice that, from the assumptions (1.4) and (1.5) on f, from Remark 2.5 and from the Scorza–Dragoni lemma, it follows that for any $\varepsilon>0$ there exists a compact set K_ε such that $\mathcal{L}^N(\Omega'\setminus K_\varepsilon)<\varepsilon$ and $a_k\big|_{K_\varepsilon\times\mathbb{R}}$ is continuous, where $a_k(x,t)=(a_{1,k}(x,t),\ldots,a_{N,k}(x,t))$. Let us now introduce the set $D_\varepsilon=\{x\in K_\varepsilon : D(K_\varepsilon;x)=1\}$ and observe that for all $t\in\mathbb{R}$ every point $x\in D_\varepsilon$ is a point of approximate continuity for $a_k(\cdot,t)$. In fact if $x\in D_\varepsilon$ we have

$$\begin{aligned}\fint_{B_\varrho(x)}|a_k(y,t)-a_k(x,t)|\mathrm{d}y&\le\frac{1}{\omega_N\varrho^N}\int_{B_\varrho(x)\cap K_\varepsilon}|a_k(y,t)-a_k(x,t)|\mathrm{d}y\\&\quad+2\sup_{y\in\Omega'}|a_k(y,t)|\frac{\mathcal{L}^N\big(B_\varrho(x)\setminus K_\varepsilon\big)}{\omega_N\varrho^N}\end{aligned}$$

and the right hand side is infinitesimal as $\varepsilon\to 0$, since $a_k(\cdot,t)$ is continuous on K_ε and K_ε has density 1 at x. Therefore, we may conclude that

$$J_{a_k(\cdot,t)}\cap D_\varepsilon=\emptyset\quad\text{for all } t\in\mathbb{R}. \tag{3.12}$$

Recalling assumption (1.7), we get immediately that the set function, defined for any Borel subset E of Ω' by setting

$$\mu(E)=\int_{\mathrm{supp}\eta}|D_x a_k(x,t)|(E)\,\mathrm{d}t,$$

is a finite Radon measure in Ω'. Therefore, for any $m\in\mathbb{N}$, we can construct a function $\psi_m\in C^1_0(\Omega')$, such that $0\le\psi_m(x)\le 1$ for all $x\in\Omega'$ and such that, denoting by H_m the compact set $H_m=\{x\in\Omega':\psi_m(x)=1\}$, the following relations hold

$$H_m\subset G\cap D_\varepsilon\subset\{\psi_m>0\},\quad \mathcal{L}^N\big(\{\psi_m>0\}\setminus H_m\big)+\mu\big(\{\psi_m>0\}\setminus H_m\big)<\frac{1}{m}. \tag{3.13}$$

Finally, let us fix a finite family $\{A_j\}_{j\in J}$ of pairwise disjoint open sets with their closures contained in Ω', denote, for any $j\in J$, by $(\varphi_{j,r})_{r\in\mathbb{N}}$ a sequence in $C^1_0(A_j)$, with $0\le\varphi_{j,r}\le 1$, and set $\eta_{j,r}(x,t)=\varphi_{j,r}(x)\eta(t)$. Since $f(x,t,\xi)\ge\sum_{j\in J}g_k(x,t,\xi)\eta_{j,r}(x,t)\psi_m(x)$, we have

$$\liminf_{n\to\infty} F(u_n,\Omega) \geq \sum_{j\in J} \liminf_{n\to\infty} \int_{\Omega'} a_{0,k}(x,u_n(x))\eta_{j,r}(x,u_n(x))\psi_m(x)\mathrm{d}x$$
$$+\sum_{j\in J} \liminf_{n\to\infty} \int_{\Omega'} \langle a_k(x,u_n(x))\eta_{j,r}(x,u_n(x)), \nabla u_n\rangle \psi_m(x)\,\mathrm{d}x. \tag{3.14}$$

Since by Remark 2.5 the functions $a_{0,k}(x,t)$ are continuous with respect to t, we have, for all $j \in J, r, m \in \mathbb{N}$,

$$\sum_{j\in J} \lim_{n\to\infty} \int_{\Omega'} a_{0,k}(x,u_n)\eta_{j,r}(x,u_n)\psi_m(x)\,\mathrm{d}x = \sum_{j\in J} \int_{\Omega'} a_{0,k}(x,u)\eta_{j,r}(x,u)\psi_m(x)\,\mathrm{d}x. \tag{3.15}$$

Notice also that $a_k(x,t)\eta_{j,r}(x,t)$ satisfy the assumptions of Theorem 1.1, for all j, r and thus

$$\lim_{n\to\infty} \int_{\Omega'} \langle a_k(x,u_n)\eta_{j,r}(x,u_n), \nabla u_n\rangle \psi_m\,\mathrm{d}x$$
$$= \lim_{n\to\infty} -\Bigg\{ \int_{\Omega'} \mathrm{d}x \int_0^{u_n(x)} \langle a_k(x,t)\eta_{j,r}(x,t), \nabla\psi_m\rangle \mathrm{d}t$$
$$+\int_{-\infty}^{+\infty} \mathrm{d}t \int_{H_m} \mathrm{sgn}(t)\chi^*_{\Omega_{u_n,t}}(x)\psi_m(x)\,\mathrm{d}\big(\mathrm{div}_x(a_k(x,t)\eta_{j,r}(x,t))\big)$$
$$+\int_{-\infty}^{+\infty} \mathrm{d}t \int_{\Omega'\setminus H_m} \mathrm{sgn}(t)\chi^*_{\Omega_{u_n,t}}(x)\psi_m(x)\,\mathrm{d}\big(\mathrm{div}_x(a_k(x,t)\eta_{j,r}(x,t))\big)\Bigg\}. \tag{3.16}$$

By the $\mathcal{L}^N$-a.e. convergence of $u_n(x) \to u(x)$ we have that

$$\lim_{n\to\infty} \int_{\Omega'} \mathrm{d}x \int_0^{u_n(x)} \langle a_k(x,t)\eta_{j,r}(x,t), \nabla\psi_m(x)\rangle \mathrm{d}t = \int_{\Omega'} \mathrm{d}x \int_0^{u(x)} \langle a_k(x,t)\eta_{j,r}(x,t), \nabla\psi_m(x)\rangle \mathrm{d}t. \tag{3.17}$$

Step 2. The last two integrals in (3.16), where the measures $D_i a_{i,k}(\cdot,t)$ appear, require a more careful analysis. To estimate the third integral on the right hand side of (3.16), we observe that

$$\left|\int_{-\infty}^{+\infty} \mathrm{d}t \int_{\Omega'\setminus H_m} \mathrm{sgn}(t)\Big[\chi^*_{\Omega_{u_n,t}}(x) - \chi^*_{\Omega_{u,t}}(x)\Big]\psi_m(x)\,\mathrm{d}\big(\mathrm{div}_x(a_k(x,t)\eta_{j,r}(x,t))\big)\right|$$
$$\leq c(N) \int_{\mathrm{supp}\eta} |D_x a_k(\cdot,t)|(\{\psi_m > 0\} \setminus H_m)\,\mathrm{d}t$$
$$+ c(N)\|\langle a_k, \nabla\eta_{j,r}\rangle\|_{L^\infty(\Omega'\times\mathrm{supp}\eta)} \mathcal{L}^1(\mathrm{supp}\eta)\mathcal{L}^N(\{\psi_m > 0\} \setminus H_m),$$

where $c(N)$ is a constant depending only on the dimension. Therefore, from this inequality and from (3.13) we get that

$$\liminf_{n\to\infty} -\int_{-\infty}^{+\infty} \mathrm{d}t \int_{\Omega'\setminus H_m} \operatorname{sgn}(t)\chi^*_{\Omega_{u_n,t}}(x)\psi_m(x)\,\mathrm{d}\big(\operatorname{div}_x(a_k(x,t)\eta_{j,r}(x,t))\big)$$
$$\geq -\int_{-\infty}^{+\infty} \mathrm{d}t \int_{\Omega'\setminus H_m} \operatorname{sgn}(t)\chi^*_{\Omega_{u,t}}(x)\psi_m(x)\,\mathrm{d}\big(\operatorname{div}_x(a_k(x,t)\eta_{j,r}(x,t))\big)$$
$$- c(N,a_k,\eta_{j,r})\frac{1}{m}, \tag{3.18}$$

for some positive constant $c(N,a_k,\eta_{j,r})$, depending only on N, a_k and $\eta_{j,r}$.

Let us now show that for $\mathcal{L}^1$-a.e. $t\in\mathbb{R}$

$$\lim_{n\to\infty}\int_{H_m} \operatorname{sgn}(t)\chi^*_{\Omega_{u_n,t}}(x)\psi_m(x)\,\mathrm{d}\big(\operatorname{div}_x(a_k(x,t)\eta_{j,r}(x,t))\big)$$
$$=\int_{H_m} \operatorname{sgn}(t)\chi^*_{\Omega_{u,t}}(x)\psi_m(x)\,\mathrm{d}\big(\operatorname{div}_x(a_k(x,t)\eta_{j,r}(x,t))\big). \tag{3.19}$$

To this aim, recalling that for $\mathcal{L}^1$-a.e. $t\in\mathbb{R}$ the level set $\{u>t\}$ is a set of finite perimeter, we are going to prove that for $\mathcal{L}^1$-a.e. $t\in\mathbb{R}$, there exists a set $G_t\subset G\cap D_\varepsilon$ such that

$$\chi^*_{\Omega_{u_n,t}}(x)\to\chi^*_{\Omega_{u,t}}(x)\quad\text{for all } x\in G_t,\quad \mathcal{L}^N(G\cap D_\varepsilon\setminus G_t)+|D_x a_k(\cdot,t)|(G\cap D_\varepsilon\setminus G_t)=0 \tag{3.20}$$

To prove (3.20) let us fix $t\in\mathbb{R}$ such that $\{u>t\}$ is a set of finite perimeter, (2.10) holds and (2.12), (2.13) hold for all $x\in\Omega\setminus N_t$, where $\mathcal{H}^{N-1}(N_t)=0$. Let us set

$$G_t = G\cap D_\varepsilon\setminus\Big(N_t\cup\{u^-\leq t\leq u^+\}\Big).$$

Let us fix $x\in G_t$. Since $x\in G$, we have in particular that x is a point of approximate continuity for u and for all functions u_n. Moreover, since $x\notin\{u^-\leq t\leq u^+\}$ and $x\notin N_t$, by (2.12) and (2.13), we have that either $\widetilde{u}(x)>t$ or $\widetilde{u}(x)<t$. In the first case, since $x\in G$, we have that $\widetilde{u}_n(x)>t$, for n large enough, hence by (2.11) $\chi^*_{\Omega_{u_n,t}}(x)=1$ and thus $\chi^*_{\Omega_{u_n,t}}(x)\to\chi^*_{\Omega_{u,t}}(x)$. The same conclusion holds also when $\widetilde{u}(x)<t$. This proves the pointwise convergence of $\chi^*_{\Omega_{u_n,t}}$ to $\chi^*_{\Omega_{u,t}}$ in the set G_t.

To prove the equality on the right hand side of (3.20), notice that, by definition

$$G\cap D_\varepsilon\setminus G_t = G\cap D_\varepsilon\cap\Big(N_t\cup\{\widetilde{u}=t\}\Big).$$

From (2.10), $\mathcal{H}^{N-1}\big(\{u^-\leq t\leq u^+\}\setminus\partial^M\{u>t\}\big)=0$ and since $\mathcal{H}^{N-1}(\partial^M\{u>t\})<\infty$, we get that $G\cap D_\varepsilon\setminus G_t$ is a set of finite $\mathcal{H}^{N-1}$ measure. Therefore, we obtain immediately that it has zero $\mathcal{L}^N$ measure. Moreover, from (3.12) we have also that $|D^j a_k(\cdot,t)|\big(G\cap D_\varepsilon\setminus G_t\big)=0$. On the other hand, since the Cantor part of the derivative of a BV function is zero on a set of $\mathcal{H}^{N-1}$ finite measure, we have also that $|D^c a_k(\cdot,t)|\big(G\cap D_\varepsilon\setminus G_t\big)=0$, hence $|Da_k(\cdot,t)|\big(G\cap D_\varepsilon\setminus G_t\big)=0$.

Since $H_m \subset G \cap D_\varepsilon$, (3.19) follows at once from (3.20) and since $\mathcal{L}^N(H_m) + \mu(H_m)$ is finite, from (3.19) we may conclude that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{-\infty}^{+\infty} \mathrm{d}t \int_{H_m} \mathrm{sgn}(t)\chi^*_{\Omega_{u_n,t}}(x)\psi_m(x)\,\mathrm{d}\big(\mathrm{div}_x(a_k(x,t)\eta_{j,r}(x,t))\big) \\
&\quad = \int_{-\infty}^{+\infty} \mathrm{d}t \int_{H_m} \mathrm{sgn}(t)\chi^*_{\Omega_{u,t}}(x)\psi_m(x)\,\mathrm{d}\big(\mathrm{div}_x(a_k(x,t)\eta_{j,r}(x,t))\big). \qquad (3.21)
\end{aligned}$$

Step 3. Putting together (3.14), (3.15), (3.16), (3.17), (3.18) and (3.21), and using Theorem 1.1 again, we obtain that for all $k, r, m \in \mathbb{N}$

$$\begin{aligned}
\liminf_{n\to\infty} F(u_n,\Omega) \ge \sum_{j\in J} \Bigg\{ &\int_{\Omega'} a_{0,k}(x,u(x))\eta_{j,r}(x,u(x))\psi_m \mathrm{d}x \\
&- \int_{\Omega'} \mathrm{d}x \int_0^{u(x)} \langle a_k(x,t)\eta_{j,r}(x,t), \nabla\psi_m\rangle \mathrm{d}t \\
&- \int_{-\infty}^{+\infty} \mathrm{d}t \int_{\Omega'} \mathrm{sgn}(t)\chi^*_{\Omega_{u,t}}(x)\psi_m(x) d\big(\mathrm{div}_x(a_k(x,t)\eta_{j,r}(x,t))\big) \\
&- c(N,a_k,\eta_{j,r})\frac{1}{m} \Bigg\} \\
= \sum_{j\in J} \Bigg\{ &\int_{\Omega'} a_{0,k}(x,u)\eta_{j,r}(x,u)\psi_m \mathrm{d}x \\
&+ \int_{\Omega'} \langle a_k(x,u)\eta_{j,r}(x,u), \nabla u\rangle \psi_m \mathrm{d}x - c(N,a_k,\eta_{j,r})\frac{1}{m} \Bigg\}.
\end{aligned}$$

Thus, letting m tend to ∞, and recalling that $\mathcal{L}^N(\Omega \setminus G) = 0$ and $\mathcal{L}^N(K_\varepsilon \setminus D_\varepsilon) = 0$ we have

$$\liminf_{n\to\infty} F(u_n,\Omega) \ge \sum_{j\in J} \int_{K_\varepsilon} \big[a_{0,k}(x,u)\eta_{j,r}(x,u) + \langle a_k(x,u)\eta_{j,r}(x,u), \nabla u\rangle\big]\,\mathrm{d}x,$$

hence, letting also ε go to zero, we finally get

$$\liminf_{n\to\infty} F(u_n,\Omega) \ge \sum_{j\in J} \int_{\Omega'} \big[a_{0,k}(x,u)\eta_{j,r}(x,u) + \langle a_k(x,u)\eta_{j,r}(x,u), \nabla u\rangle\big]\,\mathrm{d}x.$$

Recall that in Step 1 we defined $\eta_{j,r}(x,t) = \varphi_{j,r}(x)\eta(t)$. For all $j \in J$, let us now choose a sequence $\varphi_{j,r}(x)$ pointwise converging to the characteristic function of the set A_j^+, where

$$A_j^+ := \{x \in A_j : \ a_{0,k}(x,u(x)) + \langle a_k(x,u(x)), \nabla u(x)\rangle = g_k(x,u(x),\nabla u(x)) \ge 0\}.$$

Thus,

$$\liminf_{n\to\infty} F(u_n,\Omega) \geq \sum_{j\in J} \int_{A_j} \eta(u(x)) \max\{g_k(x,u(x),\nabla u(x)),0\}\mathrm{d}x$$

and, by applying Lemma 2.3, we obtain

$$\liminf_{n\to\infty} F(u_n,\Omega) \geq \int_{\Omega'} f(x,u,\nabla u)\eta(u)\mathrm{d}x.$$

Hence, the result follows letting first $\eta(t) \uparrow 1$ for any $t \in \mathbb{R}$ and then letting $\Omega' \uparrow \Omega$. $\square$

We conclude this section by a simple extension of Theorem 1.2 to a functional depending only on the absolutely continuous part of a BV function.

Theorem 3.5. *Let f satisfy the same assumption as in* Theorem 1.2. *If (u_n) is a sequence in $W^{1,1}(\Omega)$ converging in $L^1(\Omega)$ to $u \in BV(\Omega)$, then*

$$\liminf_{n\to\infty} \int_{\Omega} f(x,u_n(x),\nabla u_n(x))\,\mathrm{d}x \geq \int_{\Omega} f(x,u(x),\nabla u(x)),$$

where ∇u is the absolutely continuous part of Du.

Proof. Let (u_n) be a sequence in $W^{1,1}(\Omega)$ converging in $L^1(\Omega)$ to $u \in BV(\Omega)$. Assume, without loss of generality, that $u_n(x) \to u(x)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and set $G = \{x \in \mathcal{C}_u : \widetilde{u}_n(x) \to \widetilde{u}(x)\}$.

Let us fix an open set $\Omega' \subset\subset \Omega$ and argue exactly as in Steps 1 and 2 of the proof of Theorem 1.2 (where the assumption that u was a Sobolev function was never used). Therefore, as before, we get that

$$\begin{aligned}\liminf_{n\to\infty} F(u_n,\Omega) \geq \sum_{j\in J} \Bigg\{ &\int_{\Omega'} a_{0,k}(x,u(x))\eta_{j,r}(x,u(x))\psi_m \mathrm{d}x \\ &- \int_{\Omega'} \mathrm{d}x \int_0^{u(x)} \langle a_k(x,t)\eta_{j,r}(x,t), \nabla\psi_m\rangle \mathrm{d}t \\ &- \int_{-\infty}^{+\infty} \mathrm{d}t \int_{\Omega'} \mathrm{sgn}(t)\chi^*_{\Omega_{u,t}}(x)\psi_m(x)\mathrm{d}\big(\mathrm{div}_x(a_k(x,t)\eta_{j,r}(x,t))\big) \\ &- c(N,a_k,\eta_{j,r})\frac{1}{m} \Bigg\},\end{aligned}$$

where all the quantities appering in this inequality are defined as in the proof of Theorem 1.2. Let us now apply to the right hand side of this inequality the chain rule formula (1.2), thus getting (recall that now u is a BV function)

$$\liminf_{n\to\infty} F(u_n,\Omega) \geq \sum_{j\in J}\Bigg\{\int_{\Omega'} a_{0,k}(x,u(x))\eta_{j,r}(x,u(x))\psi_m \mathrm{d}x$$
$$+\int_{\Omega'}\langle a_k(x,u(x))\eta_{j,r}(x,u(x)),\nabla u(x)\rangle\psi_m \mathrm{d}x$$
$$+\int_{\Omega'}\langle a_k^*(x,\widetilde{u}(x))\eta_{j,r}(x,\widetilde{u}(x)),\frac{D^c u}{|D^c u|}\rangle\psi_m d|D^c u|$$
$$+\int_{J_u}\psi_m \mathrm{d}\mathcal{H}^{N-1}\int_{u^-(x)}^{u^+(x)}\langle a_k^*(x,t)\eta_{j,r}(x,t),\nu_u(x)\rangle \mathrm{d}t\Bigg\}.$$

For all $j\in J$, let us choose the sequence $(\varphi_{j,r})$ so that $\varphi_{j,r}(x)\to\chi_{D_j^+}(x)$ for $|Du|$-a.e. $x\in\Omega$, where $D_j^+=\{x\in A_j\cap\mathcal{D}_u:\ a_{0,k}(x,u(x))+\langle a_k(x,u(x)),\nabla u(x)\rangle=g_k(x,u(x),\nabla u(x))\geq 0\}$. Thus, we have

$$\liminf_{n\to\infty} F(u_n,\Omega)\geq\sum_{j\in J}\int_{A_j\cap\mathcal{D}_u}\eta(u(x))\max\{g_k(x,u(x),\nabla u(x)),0\}\,\psi_m(x)\,\mathrm{d}x.$$

From this inequality, letting $m\to\infty$ and recalling (3.13), we obtain that

$$\liminf_{n\to\infty} F(u_n,\Omega)\geq\sum_{j\in J}\int_{A_j}\eta(u(x))\max\{g_k(x,u(x),\nabla u(x)),0\}\ \mathrm{d}x$$

and from this inequality the result follows as in the proof of Theorem 1.2. □

Acknowledgments We wish to thank both the referees for carefully reading the manuscript and for their helpful comments and suggestions.

References

1. Ambrosio, L., Dal Maso, G.: A general chain rule for distributional derivatives. Proc. Amer. Math. Soc. **108**, 691–702 (1990)
2. Ambrosio, L., Fusco, N., Pallara, D.: Functions of bounded variation and free discontinuity problems. Oxford University Press, Oxford (2000)
3. Chen, G.-Q., Frid, H.: Divergence-measure fields and hyperbolic conservation laws. Arch. Rational Mech. Anal. **147**, 89–118 (1999)
4. Dal Maso, G.: Integral representation on $BV(\Omega)$ of Γ–limits of variational integrals. Man. Math. **30**, 387–416 (1980)
5. De Cicco, V.: Lower semicontinuity for certain integral functionals on $BV(\Omega)$. Boll. U.M.I. **5-B**, 291–313 (1991)
6. De Cicco, V., Leoni, G.: A chain rule in $L^1(div;\Omega)$ and its applications to lower semicontinuity. Calc. Var. Partial Differential Equations **19**(1), 23–51 (2004)
7. De Cicco, V., Fusco, N., Verde, A.: On L^1-lower semicontinuity in BV. J. Convex Anal. **12**, 173–185 (2005)
8. De Giorgi, E.: Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio a r dimensioni. Ann. Mat. Pura Appl. **36**(4), 191–213 (1954)
9. De Giorgi, E.: Teoremi di semicontinuità nel calcolo delle variazioni. Istituto Nazionale di Alta Matematica (1968–1969)
10. Federer, H.: Geometric measure theory. Springer, Berlin Heidelberg New York (1969)
11. Fonseca, I., Leoni, G.: On lower semicontinuity and relaxation. Proc. R. Soc. Edim., Sect. A, Math. **131**, 519–565 (2001)

12. Fusco, N., Giannetti, F., Verde, A.: A remark on the L^1- lower semicontinuity for integral functionals in BV, Manusc. Math. **112**, 313–323 (2003)
13. Fusco, N., Gori, M., Maggi, F.: A remark on Serrin's Theorem. NoDEA (to appear)
14. Gori, M., Marcellini, P.: An extension of the Serrin's lower semicontinuity theorem. J. Convex Anal. **9**, 475–502 (2002)
15. Gori, M., Maggi, F., Marcellini, P.: On some sharp conditions for lower semicontinuity in L^1. Diff. Int. Eq. **16**, 51–76 (2003)
16. Serrin, J.: On the definition and properties of certain variational integrals. Trans. Amer. Math. Soc. **161**, 139–167 (1961)